MTH 203: Groups and Symmetry

Homework IV

(Due on the day of the midterm)

Problems for submission

1. Let G be a group and $H < G$.
    (a) Show that the set $\bigcap_{g\in G} gHg^{-1} \lhd G$.
    (b) Consider the set $C_G(H) = \{g \in G \,|\, gh = hg, \forall h \in H\}$ called the *centralizer of H in G*. Show that $C_G(H) < G$.
    (c) Consider the set $N_G(H) = \{g \in G \,|\, gHg^{-1} = H\}$ called the *normalizer of H in G*. Show that $C_G(H) < G$.
    (d) Show that $C_G(H) \lhd N_G(H)$. Moreover, when $H \cong \mathbb{Z}_n$, show that
    $$N_G(H)/C_G(H) \cong K,$$
    where $K < \mathbb{Z}_n^{\times}$.

Problems for practice

1. Establish the assertions in 3.2 (ix), 3.3 (ii), 3.3 (v), and 3.3 (vi) of the Lesson Plan.

2. Verify the First Isomorphism Theorem for the homomomorphisms in 3.2 (ii) (f) & (g) of the Lesson Plan.

3. Classify all homomorphisms from:
    (a) $\mathbb{Z} \to \mathbb{Z}$.
    (b) $\mathbb{Z}_m \to \mathbb{Z}_n$
    (c) $\mathbb{Z}_m \to \mathbb{Z}$
    (d) $\mathbb{Z} \to \mathbb{Z}_n$

4. Let $G = D_{2n}$, for $n \geq 3$. Let $H = \langle r^k \rangle$, for $1 \leq k \leq n-1$, and let $K = \langle s \rangle$, where s is any reflection.
    (a) Is $H, K \lhd G$? Explain why, or why not.
    (b) Compute $Z(G)$.
    (c) Compute $N_G(H)$, $N_G(K)$, $C_G(H)$, and $C_G(K)$.

5. Let G be a group, $H < G$, and $N \lhd G$. Then show that:
    (a) $NH < G$. (NH is called the *internal direct product* of N and H.)
    (b) $H \cap N \lhd H$.
    (c) $N \lhd HN$.
    (d) If $H \lhd G$, then $NH \lhd G$.
    (e) If $o(a)$ is finite for some $a \in G$, then $o(Na) \mid o(a)$.

6. Let G be a group, and $H < G$. Then prove that:

(a) $N_G(H) < G$.

(b) $H \lhd N_G(H)$.

(c) $N_G(H)$ is the largest subgroup in which H is normal.

(d) $H \lhd G$ if, and only if $N_G(H) = G$.

7. For a group G, consider the subset

$$[G, G] = \{ghg^{-1}h^{-1} : g, h \in G.\}$$

(a) Show that $[G, G] \lhd G$.

(b) Show that $G/[G, G]$ is abelian.